# Voronoi Cells in Metric Algebraic Geometry of Plane Curves

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# Overview

1. Convergence of Voronoi diagrams of point clouds to Voronoi cells of varieties

2. Application: medial axis

3. Application: curvature and the evolute

# Voronoi Diagrams of Point Clouds

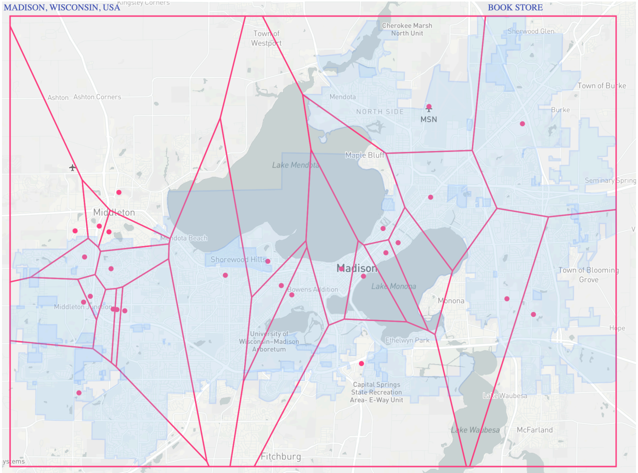


Figure: A Voronoi diagram of the bookstores in Madison. Created using program by Rodion Chachura.

# Voronoi Cells of Varieties

## Definition

Let $X$ be a real algebraic variety of codimension $c$ in $\mathbb{R}^n$ and $y$ a smooth point on $X$. Its *Voronoi cell* consists of all points whose closest point in $X$ is $y$, i.e.

$$\mathrm{Vor}_X(y) \ := \ \left\{ u \in \mathbb{R}^n : y \in \operatorname*{arg\,min}_{x \in X} \|x - u\|^2 \right\}.$$

The Voronoi cell $\mathrm{Vor}_X(y)$ is a convex semialgebraic set of dimension $c$, living in the normal space $N_X(y)$ to $X$ at $y$. Its boundary consists of the points in $\mathbb{R}^n$ that have at least two closest points in $X$, including $y$.

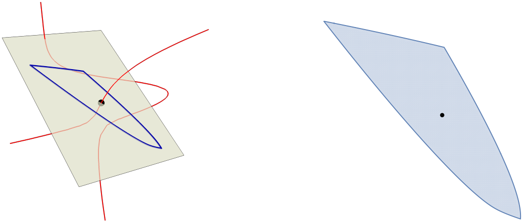

Figure: A quartic space curve and a Voronoi cell in a normal plane.

# Convergence Theorem: Voronoi Version

## Definition

Given a point $x \in \mathbb{R}^n$ and a closed set $B \subset \mathbb{R}^n$, define

$$d_w(x, B) = \inf_{b \in B} d(x, b).$$

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Wijsman convergent* to $B$ if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, B_\nu) \to d_w(x, B).$$

## Theorem (Brandt-W. '19)

*Let $X$ be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of $X$ containing all singular points of $X$ such that no point of $X$ is more than distance $\epsilon$ from every point in $A_\epsilon$. Then every Voronoi cell of $X$ is the Wijsman limit of a sequence of Voronoi cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.*

# Example of Convergence Theorem

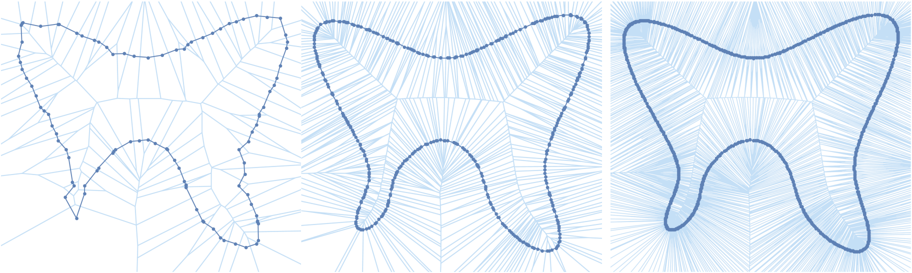

Figure: Voronoi cells of 101, 441, and 1179 points sampled from the quartic butterfly curve: $x^4 - x^2y^2 + y^4 - 4x^2 - 2y^2 - x - 4y + 1 = 0$.

# Definition of Medial Axis

**Definition**

The *medial axis* of a variety $X$ is the locus of points in the ambient space which have more than one nearest point on $X$.

If $X$ is a finite point set, then its medial axis is the Voronoi diagram. For a general variety, the medial axis is composed of the boundaries of Voronoi cells.

# Long Edges and Short Edges

## Proposition/Definition (Brandt '94)

*Let $X \subset \mathbb{R}^2$ be a compact smooth plane curve, and let $A_\epsilon$ be an $\epsilon$-approximation of $X$. A Voronoi cell $Vor_{A_\epsilon}(a_\epsilon)$ for $a_\epsilon \in A_\epsilon$ is polyhedral, meaning it is an intersection of half-spaces. For sufficiently small $\epsilon$, exactly two edges of $Vor_{A_\epsilon}(a_\epsilon)$ will intersect $X$. We call these edges the* **long edges** *of the Voronoi cell, and all other edges are called* **short edges**.

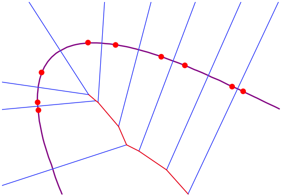

Figure: The long edges (blue) and short edges (red) of Voronoi cells of points sampled from the butterfly curve.

# Medial Axis from Voronoi Cells

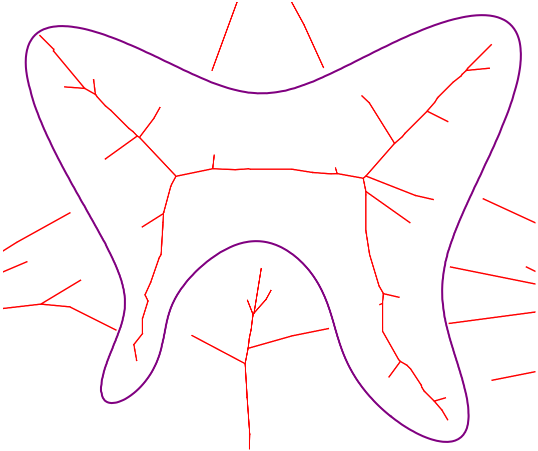

Figure: A medial axis approximation of the butterfly curve obtained from short edges of Voronoi cells, which are shown in red.

# Definition 1 of Curvature: Using "Infinitely Close" Points

## Definition (Cauchy, 1826)

Let $X \subset \mathbb{R}^2$ be a curve and $p \in X$ a smooth point. The *center of curvature* at $p$ is the intersection of the normal line to $X$ at p and the normal line to $X$ at a point **infinitely close** to $p$. The *radius of curvature* at $p$ is the distance from $p$ to its center of curvature. The (unsigned) *curvature* is the reciprocal of the radius of curvature.

# Definition 2 of Curvature: Using Envelope of Normal Lines

## Definition

The *envelope* of a one-parameter family of plane curves given implicitly by $F(x, y, t) = 0$ is a curve that touches every member of the family tangentially. The envelope is the variety defined by the ideal

$$\left\langle \frac{\partial F}{\partial t},\ F(x, y, t) \right\rangle \subset \mathbb{R}[x, y].$$

The envelope of the family of normal lines parametrized by the points of the curve is called its *evolute*. The evolute is the locus of the *centers of curvature.*

# Curvature and the Evolute

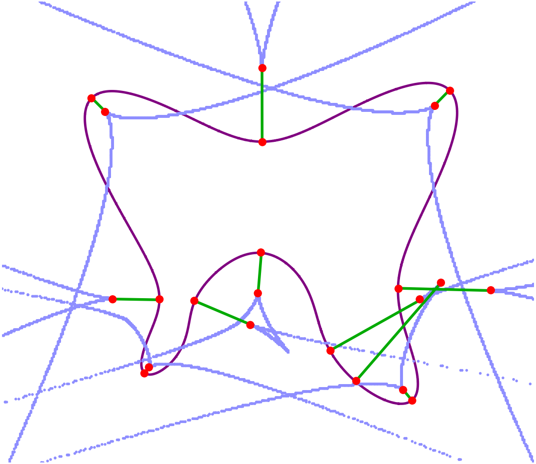

Figure: The eleven real points of critical curvature on the butterfly curve (purple) joined by green line segments to their centers of curvature. These give cusps on the evolute (light blue).

# Voronoi Methods for Computing Curvature

**Theorem (Brandt-W. '19)**

*Let $X$ be a smooth plane curve of degree at least 3. Fix $p \in X$. Let $\delta$ be less than the minimum of the reach and the distance to the nearest critical point of curvature to $p$, and let $B(p, \delta)$ be a ball of radius $\delta$ centered at $p$. Then*

1. *The Voronoi cell $Vor_{X \cap B(p,\delta)}(p)$ is a ray. The distance from $p$ to the endpoint of this ray is the radius of curvature of $X$ at $p$.*
2. *Consider a sequence of $\epsilon$-approximations $A_\epsilon$ of $X \cap B_{p,\delta}$. Let $a_\epsilon$ be the point such that $p \in Vor_{A_\epsilon}(a_\epsilon)$, and let $d_\epsilon$ be the minimum distance from $a_\epsilon$ to a vertex of $Vor_{A_\epsilon}(a_\epsilon)$. Then the sequence $d_\epsilon$ converges to the radius of curvature of $p$.*

# Degree of Critical Curvature

**Theorem (Brandt-W. '19)**

*Let $X \subset \mathbb{R}^2$ be a smooth, irreducible curve of degree $d \geq 3$. Then the degree of critical curvature of $X$ is $6d^2 - 10d$.*

# Current Work

- Study the degree of critical curvature of varieties of higher dimension.
- Use exact methods to study metric algebraic geometry of varieties.
  - reach
  - curvature
  - bottlenecks
  - Voronoi decomposition
  - Euclidean Distance Degree and Discriminant

Thank you!

# Convergence Theorem: Delaunay Version

**Theorem (Brandt-W. '19)**

*Let $X$ be a compact curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ a sequence of finite subsets of $X$ containing all singular points of $X$ such that no point of $X$ is more than distance $\epsilon$ from every point in $A_\epsilon$. If $X$ is not tangent to any circle in four or more points, then every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of $\{A_\epsilon\}_{\epsilon \searrow 0}$.*

# Delaunay Cells

## Definition

Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to $X$ if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Given an inscribed disc $B$ of an algebraic variety $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap X)$.

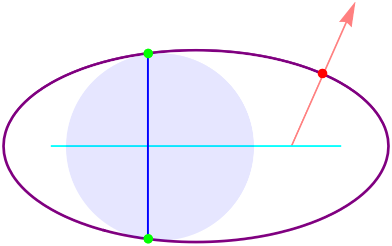

Figure: The dark blue line segment is a Delaunay cell defined by the light blue maximally inscribed circle with center $(-3/8, 0)$ and radius $\sqrt{61}/8$.

# Duality of Delaunay and Voronoi Cells

## Definition

Let $X \subset \mathbb{R}^2$ be a finite point set. A *Delaunay triangulation* is a triangulation $DT(X)$ of $X$ such that no point of $X$ is inside the circumcircle of any triangle of $DT(X)$.

## Remark

*The circumcenters of triangles in $DT(X)$ are the vertices in the Voronoi diagram of $X$.*

# Hausdorff Convergence

The *Hausdorff distance* of two compact sets $B_1$ and $B_2$ in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) := \sup \left\{ \sup_{x \in B_1} \inf_{y \in B_2} d(x, y), \sup_{y \in B_2} \inf_{x \in B_1} d(x, y) \right\}.$$

If an adversary gets to put your ice cream on either set $B_1$ or $B_2$ with the goal of making you go as far as possible, and you get to pick your starting place in the opposite set, then $d_h(B_1, B_2)$ is the farthest the adversary could make you walk in order for you to reach your ice cream.

## Definition

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Hausdorff convergent* to $B$ if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$.