Voronoi Cells of Varieties

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Overview

Voronoi cells

Definition

Let X be a real algebraic variety of codimension c in $\mathbb{R}^n$ and y a smooth point on X. Its *Voronoi cell* consists of all points whose closest point in X is y, i.e.

$$\mathrm{Vor}_X(y) \;:=\; \left\{\, u \in \mathbb{R}^n : y \in \underset{x \in X}{\arg\min}\, \|x - u\|^2 \,\right\}.$$

The Voronoi cell $\mathrm{Vor}_X(y)$ is a convex semialgebraic set of dimension c, living in the normal space $N_X(y)$ to X at y. Its boundary consists of the points in $\mathbb{R}^n$ that have at least two closest points in X, including y.

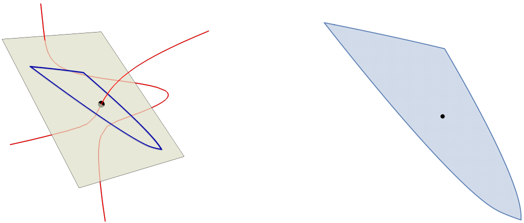

Figure: A quartic space curve and the Voronoi cell in one of its normal planes.

Voronoi ideal

Let $I = \langle f_1, f_2, \ldots, f_m \rangle$ in $\mathbb{Q}[x_1, \ldots, x_n]$ and $X = V(I) \subset \mathbb{R}^n$. We will now give equations for the ideal of $\mathrm{Vor}_X(y)$. Consider the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_n, u_1, \ldots, u_n]$ where $u = (u_1, \ldots, u_n)$ is an additional unknown point.

Definition

The *augmented Jacobian* of X at x is

$$J_I(x, u) \quad := \quad \begin{bmatrix} u - x \\ (\nabla f_1)(x) \\ \vdots \\ (\nabla f_m)(x) \end{bmatrix}$$

Voronoi ideal

Let N_I denote the ideal in R generated by I and the $(c+1) \times (c+1)$ minors of the augmented Jacobian $J_I(x,u)$, where c is the codimension of the given variety $X \subset \mathbb{R}^n$. The ideal N_I in R defines a subvariety of dimension n in $\mathbb{R}^{2n}$, namely the *Euclidean normal bundle* of X. Let $N_I(y)$ denote the linear ideal that is obtained from N_I by replacing the unknown point x by the given point $y \in \mathbb{R}^n$.

Example

Let $n = 2$ and $I = \langle\, x_1^3 - x_2^2 \,\rangle$, so $X = V(I) \subset \mathbb{R}^2$ is a cubic curve with a cusp at the origin. The ideal of the Euclidean normal bundle of X is

$$N_I \;=\; \left\langle\, x_1^3 - x_2^2\,,\, \det\begin{pmatrix} u_1 - x_1 & u_2 - x_2 \\ 3x_1^2 & -2x_2 \end{pmatrix} \right\rangle.$$

For the point $y = (4,8)$, we have $N_I(y) = \langle u_1 + 3u_2 - 28 \rangle$.

Voronoi ideal

Let $I = \langle f_1, f_2, \ldots, f_m \rangle$ in $\mathbb{Q}[x_1, \ldots, x_n]$ and $X = V(I) \subset \mathbb{R}^n$.

Definition

The *critical ideal* of the variety X at the point y is

$$C_I(y) \quad = \quad I + N_I + N_I(y) + \langle \, \|x - u\|^2 - \|y - u\|^2 \rangle.$$

Definition

The *Voronoi ideal* is the following ideal in $\mathbb{Q}[u_1, \ldots, u_n]$. It is obtained from the critical ideal by saturation and elimination:

$$\mathrm{Vor}_I(y) \quad = \quad \big(\, C_I(y) \, : \, \langle x - y \rangle^\infty \big) \cap \mathbb{Q}[u_1, \ldots, u_n].$$

Voronoi ideal

Example

Let $n = 2$ and $I = \langle\, x_1^3 - x_2^2 \,\rangle$, so $X = V(I) \subset \mathbb{R}^2$ is a cubic curve with a cusp at the origin. Let $y = (4, 8)$.
The Voronoi ideal is

$$\mathrm{Vor}_I(y) \;\; =$$

$$\langle u_1 - 28, u_2\rangle \;\cap\; \langle u_1 + 26, u_2 - 18\rangle \;\cap\; \langle u_1 + 3u_2 - 28,\, 27u_2^2 - 486u_2 + 2197\rangle.$$

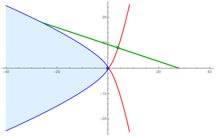

Figure: The cuspidal cubic is shown in red. The Voronoi cell of a smooth point is a green line segment. The Voronoi cell of the cusp is the convex region bounded by the blue curve.

Voronoi degree

Remark

When discussing degree, we identify the variety X and $\delta_{alg}\mathrm{Vor}_X(y)$ with their Zariski closures in $\mathbb{P}^n_{\mathbb{C}}$.

Definition

The algebraic boundary of the Voronoi cell $\mathrm{Vor}_X(y)$ is a hypersurface in the normal space to X at y. Its degree $\delta_X(y)$ is called the *Voronoi degree* of X at y.

Voronoi degree of curve

Theorem

Let $X \subset \mathbb{P}^n$ be a curve of degree d and geometric genus g with at most ordinary multiple points as singularities. The Voronoi degree at a general point $y \in X$ equals

$$\delta_X(y) \ = \ 4d + 2g - 6,$$

provided X is in general position in $\mathbb{P}^n$.

Voronoi degree of curve

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provided X is in general position in $\mathbb{P}^n$.

Example

If X is a rational curve of degree d, then $g = 0$ and hence $\delta_X(y) = 4d - 6$. If X is an elliptic curve, so the genus is $g = 1$, then we have $\delta_X(y) = 4d - 4$.

Voronoi degree of curve: Example

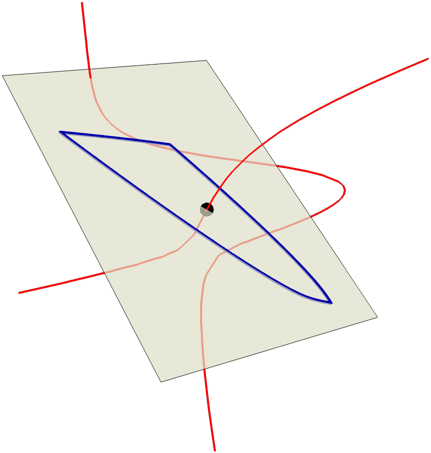


Figure: A space curve with $d = 4$ and $g = 1$ has Voronoi degree $\delta_X(y) = 12$.

Voronoi degree of surface

Theorem

Let $X \subset \mathbb{P}^n$ be a smooth surface of degree d. Then its Voronoi degree equals

$$\delta_X(y) \;=\; 3d + \chi(X) + 4g(X) - 11,$$

provided the surface X is in general position in $\mathbb{P}^n$ and y is a general point on X.

- $\chi(X) := c_2(X)$ is the topological Euler characteristic, which equals the degree of the second Chern class of the tangent bundle
- $g(X)$ is the genus of the curve obtained by intersecting X with a general smooth quadratic hypersurface in $\mathbb{P}^n$

Norms on Space of Matrices

Fix the space $\mathbb{R}^{m \times n}$ of real $m \times n$ matrices.

Frobenius Norm

$\|U\|_F := \sqrt{\sum_{ij} U_{ij}^2}$

Spectral Norm

$\|U\|_2 := \max_i \sigma_i(U)$ which extracts the largest singular value.

Low Rank Matrix Approximation

Remark

Let X denote the variety in $\mathbb{R}^{m\times n}$ of real $m \times n$ matrices of rank $\leq r$. Fix a rank r matrix V in X. Let $U \in \mathrm{Vor}_X(V)$ and let $U = \Sigma_1 D \Sigma_2$ be its singular value decomposition. Let $D^{[r]}$ be the matrix that is obtained from D by replacing all singular values except for the r largest ones by zero. By the Eckart-Young Theorem, we have $V = \Sigma_1 \cdot D^{[r]} \cdot \Sigma_2$.

Remark

The Eckart-Young Theorem works for both norms, so both give the same Voronoi cell $\mathrm{Vor}_X(V)$.

Voronoi cell of low rank matrix

Theorem

Let V be an $m \times n$-matrix of rank r. The Voronoi cell $\mathrm{Vor}_X(V)$ is congruent up to scaling to the unit ball in the spectral norm on the space of $(m-r) \times (n-r)$-matrices.

Norms on Space of Symmetric Matrices

Consider the space $\mathbb{R}^{\binom{n+1}{2}}$ whose coordinates are the upper triangular entries of a symmetric $n \times n$ matrix. Let X be the variety of symmetric matrices of rank $\leq r$.

Remark

The Frobenius norm and Euclidean norm differ on this space.

The Frobenius norm on $\mathbb{R}^{\binom{n+1}{2}}$ is the restriction of the Frobenius norm on $\mathbb{R}^{n \times n}$ to the subspace of symmetric matrices.

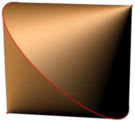
Example

Let $n = 2$. We identify the vector (a, b, c) with the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The Frobenius norm is $\sqrt{a^2+2b^2+c^2}$, whereas the Euclidean norm is $\sqrt{a^2+b^2+c^2}$.

Low Rank Approximation on the Space of Symmetric Matrices

Remark

The Frobenius and Euclidean norms have dramatically different properties with respect to low rank approximation. The Eckart-Young Theorem remains valid for the Frobenius norm on $\mathbb{R}^{\binom{n+1}{2}}$*, but not for the Euclidean norm.*



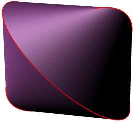


Figure: The Voronoi cell of a symmetric 3×3 matrix of rank 1 is a convex body of dimension 3. It is shown for the Frobenius norm (left) and for the Euclidean norm (right).

New Work: Voronoi Cells in Metric Algebraic Geometry of Plane Curves

Joint with Madeline Brandt, Arxiv 1906.11337

- Study limits of Voronoi decompositions.
- Detect features of variety (reach, medial axis, bottlenecks, maximal radius of curvature) from Voronoi decomposition.

Thank you!