Algebraic Geometry of Curvature and Matrices with Partitioned Eigenvalues

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The Reach of an Algebraic Variety

Definition

The **medial axis** of a variety $V \subset \mathbb{R}^n$ is the set $Med(V)$ of all points $u \in \mathbb{R}^n$ such that the minimum distance from V to u is attained by two distinct points. The **reach** τ_V is the infimum of all distances from points on the variety V to points in its medial axis $Med(V)$.

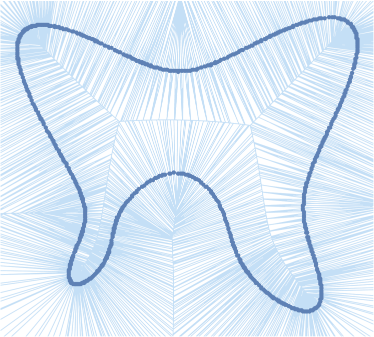


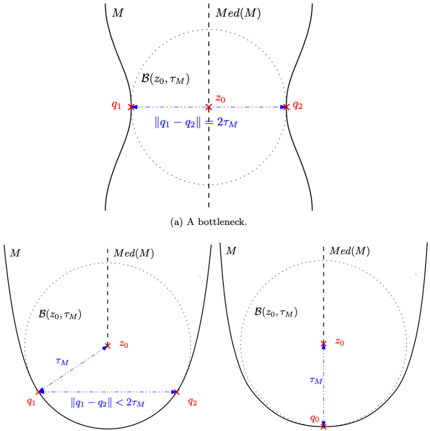
Figure: The medial axis of the quartic butterfly curve can be seen in its Voronoi approximation.

Algebraicity of Reach

Proposition (Horobet-W. '18)

Let V be a smooth algebraic variety in $\mathbb{R}^n$. Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ with $V = V_{\mathbb{R}}(f_1, \ldots, f_s)$. Then the reach of V is an algebraic number over $\mathbb{Q}$.

Reach, Bottlenecks, and Curvature



(a) A bottleneck.

(b) A non-bottleneck reach attaining pair.

(c) Curvature-attaining point.

Figure: The reach of a manifold is attained by a bottleneck, two points on a circular arc, or a point of maximal curvature. Figure and Theorem due to Aamari-Kim-Chazal-Michel-Rinaldo-Wasserman '17.

Bottleneck Degree

Denote by $BND(V)$ the bottleneck degree of $V \subset \mathbb{C}^n$. Under certain conditions, this coincides with twice the number of bottleneck pairs.

Theorem (Di Rocco-Eklund-W. '19)

- *Let $V \subset \mathbb{C}^2$ be a "general" curve of degree d. Then $BND(V) = d^4 - 5d^2 + 4d$.*
- *Let $V \subset \mathbb{C}^3$ be a "general" surface of degree d. Then $BND(V) = d^6 - 2d^5 + 3d^4 - 15d^3 + 26d^2 - 13d$.*
- *For any smooth variety $V \subset \mathbb{P}^n_{\mathbb{C}}$ in "general position," we have an algorithm to express the bottleneck degree in terms of the polar classes of V.*

Building Bridges Between Differential Geometry and Computational Algebraic Geometry

- Curvature is central to the study of differential geometry.
- Curvature is a property of algebraic varieties.
- Properties of algebraic varieties should have defining polynomial equations and degrees!

Algebraic Manifold: Algebraic Variety and Differentiable Manifold

- $f \in \mathbb{R}[x_1, \ldots, x_n]$
- $V = \{x \in \mathbb{C}^n | f(x) = 0\}$ smooth algebraic variety
- $M = V \cap \mathbb{R}^n$ differentiable submanifold of $\mathbb{R}^n$
- M is an **algebraic manifold**

Euclidean Connection and Second Fundamental Form

For any manifold M, let $\mathcal{T}(M)$ denote the set of smooth vector fields on M; this is the space of smooth sections of the tangent bundle TM. For $M \subset \mathbb{R}^n$, let $\mathcal{N}(M)$ denote the space of smooth sections of the normal bundle NM. The **Euclidean connection** $\overline{\nabla}$ on $\mathbb{R}^n$ is a map $\overline{\nabla} : \mathcal{T}(\mathbb{R}^n) \times \mathcal{T}(\mathbb{R}^n) \to \mathcal{T}(\mathbb{R}^n), (X, Y) \mapsto \overline{\nabla}_X Y$ defined as follows:

$$(\overline{\nabla}_X Y)(p) = \sum_{i=1}^{n} X_i(p) \frac{\partial Y}{\partial x_i}(p).$$

In other words, $\overline{\nabla}_X Y$ is the vector field whose components are the directional derivatives of the components of Y in the direction X.
The **second fundamental form** of M is the map II from $\mathcal{T}(M) \times \mathcal{T}(M)$ to $\mathcal{N}(M)$ given by

$$\mathrm{II}(X, Y) := (\overline{\nabla}_X Y)^{\perp}.$$

Principal Curvatures

Let $M \subset \mathbb{R}^3$ be a surface. Fix a point $p \in M$ and vector fields $X, Y \in \mathcal{T}(M)$ such that $X(p)$ and $Y(p)$ form an orthonormal basis of T_pM. Let $N(p)$ be a unit vector in N_pM. The **principal curvatures** of M at p are the eigenvalues of the symmetric matrix

$$\begin{bmatrix} \mathrm{II}(X,X)(p)\cdot N(p) & \mathrm{II}(X,Y)(p)\cdot N(p) \\ \mathrm{II}(X,Y)(p)\cdot N(p) & \mathrm{II}(Y,Y)(p)\cdot N(p) \end{bmatrix}.$$

If X and Y are selected so that the matrix is diagonal, then $X(p)$ and $Y(p)$ are the **principal directions**, up to a choice of normal vector.

Critical Curvature Points and Umbilics

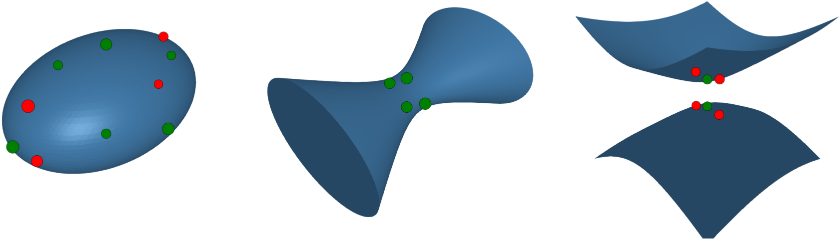


Figure: The pictures show the three quadric surfaces $X_1 = \{x_1^2 + 2x_2^2 + 4x_3^2 = 1\}$ (left picture) and $X_2 = \{-x_1^2 + 2x_2^2 + 4x_3^2 = 1\}$ (picture in the middle) and $X_3 = \{-2x_1^2 - x_2^2 + 4x_3^2 = 1\}$ (right picture). The critical curvature points are shown in green and the umbilical points are shown in red.

Degree of Critical Curvature

Theorem (Brandt-W. '19)

Let $V \subset \mathbb{R}^2$ be a smooth, irreducible curve of degree $d \geq 3$. Then the degree of critical curvature of V is $6d^2 - 10d$.

Equations for Critical Curvature Locus

The following equations define the locus of pairs (x, u) where $x \in M$ and u is a principal direction at x:

$$f(x_1, \ldots, x_n) = 0,$$

$$\nabla f \cdot u = 0,$$

$$\sum_{i=1}^{n} u_i^2 - 1 = 0,$$

$$\lambda^2(\nabla f \cdot \nabla f) - 1 = 0,$$

$$H_f \cdot u + y_1 u + y_2 \nabla f = 0.$$

The curvature is given by the absolute value of $g(x, u, \lambda) = \lambda u^t \cdot H_f \cdot u$. Using the principle of Lagrange multipliers, we intersect the above locus with the locus defined by the vanishing of the minors of a matrix of partial derivatives of the above equations and partial derivatives of g.

Upper Bound for Critical Curvature Degree

Theorem (Breiding-Ranestad-W.'21)

Let $V \subset \mathbb{R}^3$ be a smooth, irreducible surface of degree $d \geq 3$. There are only finitely many complex critical curvature points of V. An upper bound for their number is given by $\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$.

d	$\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$	actual number
2	498	18
3	3573	≥ 456
4	11328	≥ 1808

Umbilics

Theorem (Salmon 1865)

The degree of the variety of umbilics of a general surface of degree d in $\mathbb{R}^3$ is $10d^3 - 28d^2 + 22d$.

From Umbilics to Matrices with Partitioned Eigenvalues

Umbilics occur when the matrix of the second fundamental form has repeated eigenvalues. What is known about the algebraic geometry of matrices with repeated eigenvalues?

Definition of Variety of Matrices with Partitioned Eigenvalues

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. Let $\mathbb{R}^{\frac{n(n+1)}{2}}$ be the space of real symmetric $n \times n$ matrices. The **variety of λ-partitioned eigenvalues** $V_{\mathbb{R}}(\lambda) \subset \mathbb{R}^{\frac{n(n+1)}{2}}$ is the Zariski closure of the locus of matrices with eigenvalue multiplicities determined by λ.

Real Symmetric vs. Real Square, Complex Symmetric, Complex Square

This table shows the dimension of the locus of matrices with a given Jordan normal form.

	$\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$	$\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$
Complex Square	1	3
Complex Symmetric	1	2
Real Symmetric	1	Empty

Key Idea

Real symmetric matrices can be studied through their diagonalizations.

Dimension

Theorem (W. '20)

The complexification $V_{\mathbb{C}}(\lambda)$ of the real algebraic variety $V_{\mathbb{R}}(\lambda) \subset \mathbb{R}^{\frac{n(n+1)}{2}}$ of $n \times n$ real symmetric matrices with eigenvalue multiplicities corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n or partitions coarser than λ is an irreducible variety of dimension $m + \binom{n}{2} - \sum_{i=1}^{m} \binom{\lambda_i}{2}$.

Parametrization

Proposition (W. '20)

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n such that $\lambda \neq (1, \ldots, 1)$. Let $Diag(\lambda)$ be a diagonal $n \times n$ matrix with diagonal entries $\mu_1, \ldots, \mu_m$ where each entry μ_i appears with multiplicity λ_i. Let B be a skew-symmetric $n \times n$ matrix. Let I be the $n \times n$ identity matrix. The map

$$p : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$$

$$B \mapsto (I - B)(I + B)^{-1} Diag(\lambda)(I + B)(I - B)^{-1}$$

is a parametrization of a Zariski open dense subset of $V_{\mathbb{R}}(\lambda)$ by rational functions.

Equations: for small n

We use the parametrization to generate points on the variety and then use interpolation to find polynomials that vanish on these points.

Example

The ideal $I(V_{\mathbb{R}}((2,2)))$ is of codimension 4 and degree 6. It is generated by the following 9 quadrics:

$$
\begin{gathered}
x_{11}^2 + 4x_{13}^2 - x_{22}^2 - 4x_{24}^2 - 2x_{11}x_{33} + x_{33}^2 + 2x_{22}x_{44} - x_{44}^2 \\
x_{11}x_{12} + x_{12}x_{22} + 2x_{13}x_{23} + 2x_{14}x_{24} - x_{12}x_{33} - x_{12}x_{44} \\
x_{11}x_{14} - x_{14}x_{22} + 2x_{12}x_{24} - x_{14}x_{33} + 2x_{13}x_{34} + x_{14}x_{44} \\
x_{11}x_{13} - x_{13}x_{22} + 2x_{12}x_{23} + x_{13}x_{33} + 2x_{14}x_{34} - x_{13}x_{44} \\
-x_{11}^2 - 4x_{14}^2 + x_{22}^2 + 4x_{23}^2 - 2x_{22}x_{33} + x_{33}^2 + 2x_{11}x_{44} - x_{44}^2 \\
2x_{12}x_{14} - x_{11}x_{24} + x_{22}x_{24} - x_{24}x_{33} + 2x_{23}x_{34} + x_{24}x_{44} \\
2x_{12}x_{13} - x_{11}x_{23} + x_{22}x_{23} + x_{23}x_{33} + 2x_{24}x_{34} - x_{23}x_{44} \\
-x_{11}^2 - 4x_{12}^2 + 2x_{11}x_{22} - x_{22}^2 + x_{33}^2 + 4x_{34}^2 - 2x_{33}x_{44} + x_{44}^2 \\
-x_{11}x_{34} + 2x_{13}x_{14} - x_{22}x_{34} + 2x_{23}x_{24} + x_{33}x_{34} + x_{34}x_{44}
\end{gathered}
$$

Equations: for large n

The ideal $I(V_{\mathbb{R}}(\lambda))$ is stable under the action by conjugation of the real orthogonal group $O(n)$ on the space $\mathbb{R}^{\frac{n(n+1)}{2}}$ of real symmetric $n \times n$-matrices. Thus the degree d homogeneous component $I(V_{\mathbb{R}}(\lambda))_d$ is a representation of $O(n)$.

Invariants under Action by Orthogonal Group

Denote by $I(V_{\mathbb{R}}(\lambda))^{O(n)}$ the graded vector space of $O(n)$-invariant polynomials in $I(V_{\mathbb{R}}(\lambda))$. Let $V_{\mathbb{R}}(D_\lambda)$ denote the intersection of $V_{\mathbb{R}}(\lambda)$ with the variety of diagonal matrices in $\mathbb{R}^{\frac{n(n+1)}{2}}$. The symmetric group $S_n \subset O(n)$, consisting of the permutation matrices, acts on $V_{\mathbb{R}}(D_\lambda)$ by permuting the diagonal entries. Let $I(V_{\mathbb{R}}(D_\lambda))^{S_n}$ be the graded vector space of S_n-invariant polynomials in $I(V_{\mathbb{R}}(D_\lambda))$.

Theorem (W. '20)

$I(V_{\mathbb{R}}(\lambda))^{O(n)}$ and $I(V_{\mathbb{R}}(D_\lambda))^{S_n}$ are isomorphic as graded vector spaces.

Diagonal Variety and Euclidean Distance Degree

Proposition

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. The degree of the variety $V_{\mathbb{R}}(D_\lambda)$ of $n \times n$ diagonal matrices with eigenvalue multiplicities partitioned according to λ is

$$\frac{n!}{\lambda_1! \cdots \lambda_m!}$$

Theorem (Bik and Draisma)

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$. The Euclidean distance degree of the variety $V_{\mathbb{R}}(\lambda)$ of λ-partitioned eigenvalues is $\frac{n!}{\lambda_1! \cdots \lambda_m!}$.

Future Work

- Obtain an exact formula, or tighter bound, for the critical curvature degree.
- Formulate systems of polynomial equations for other concepts in differential geometry and distance optimization.

Thank you!

Polar Classes

Example

For a smooth surface $V \subset \mathbb{P}^3$ we have two polar varieties. Let $p \in \mathbb{P}^3$ be a general point and $l \subset \mathbb{P}^3$ a general line. Then $P_1(V, p)$ is the set of points $x \in V$ such that the projective tangent plane $\mathbb{T}_x V \subset \mathbb{P}^3$ contains p. This is a curve on V. Similarly, $P_2(V, l) = \{x \in V : l \subseteq \mathbb{T}_x V\}$, which is finite.

Definition

Let $V \subset \mathbb{P}^n$ be a smooth variety of dimension m. For $j = 0, \ldots, m$ and a general linear space $L \subseteq \mathbb{P}^n$ of dimension $n - m - 2 + j$ the **polar variety** is given by

$$P_j(V, L) = \{x \in V : \dim \mathbb{T}_x V \cap L \geq j - 1\}.$$

For each polar variety $P_j(V, L)$, there is a corresponding **polar class** $[P_j(V, L)] = p_j$ which represents $P_j(V, L)$ up to rational equivalence.

$P_j(V, L)$ is either empty or of pure codimension j and

$$p_j = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} c_i(T_X),$$

where $h \in A_{n-1}(X)$ is the hyperplane class.
The polar loci $P_j(V, L)$ are reduced. We have

$$c_j(T_X) = \sum_{i=0}^{j} (-1)^i \binom{m-i+1}{j-i} h^{j-i} p_i.$$

Bottleneck Genericity Assumptions 1/2

Let $V \subset \mathbb{P}^n$ be a variety. Consider the **conormal variety**

$$\mathcal{C}_V = \{(p, q) \in \mathbb{P}^n \times \mathbb{P}^n : p \in V, q \in (\mathbb{T}_p V)^\perp\}$$

and the map

$$f : \mathcal{C}_V \to \mathrm{Gr}(2, n+1) : (p, q) \mapsto \langle p, q \rangle$$

from $\mathcal{C}_V$ to the Grassmannian of lines in $\mathbb{P}^n$ that sends a pair (p, q) to the line spanned by p and q.
The orthogonality relation on $\mathbb{P}^n$ is defined via the **isotropic quadric** $Q \subset \mathbb{P}^n$ given in homogeneous coordinates by $\sum_0^n x_i^2 = 0$. Varieties which are tangent to Q are to be considered degenerate in this context and we say that a smooth projective variety is in **general position** if it intersects Q transversely. Equivalently, a smooth variety $V \subset \mathbb{P}^n$ is in general position if $\mathcal{C}_V$ is disjoint from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

Bottleneck Genericity Assumptions 2/2

A smooth variety $V \subset \mathbb{P}^n$ is **bottleneck regular** if

1. V is in general position,
2. V has only finitely many bottlenecks and
3. the differential $df_p : T_p\mathcal{C}_V \to T_{f(p)}G$ of the map f has full rank for all $p \in \mathcal{C}_V$.

If $V \subset \mathbb{P}^n$ is bottleneck regular, then V is equal to the number of bottlenecks of V counted with multiplicity.