Enumerative Geometry of Curvature

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Building Bridges Between Differential Geometry and Computational Algebraic Geometry

- Curvature is central to the study of differential geometry.
- Curvature is a property of algebraic varieties.
- Properties of algebraic varieties should have defining polynomial equations and degrees!

Curvature and the Evolute

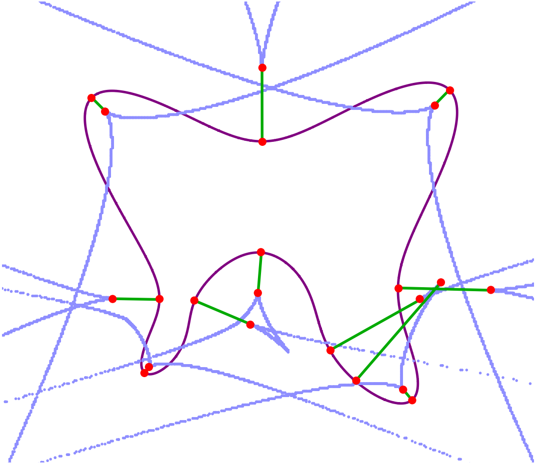

Figure: The eleven real points of critical curvature on the butterfly curve (purple) joined by green line segments to their centers of curvature. These give cusps on the evolute (light blue).

Degree of Critical Curvature

Theorem (Brandt-W.)

Let $V \subset \mathbb{R}^2$ be a smooth, irreducible curve of degree $d \geq 3$. Then the degree of critical curvature of V is $6d^2 - 10d$.

Algebraic Manifold: Algebraic Variety and Differentiable Manifold

- $f \in \mathbb{R}[x_1, \ldots, x_n]$
- $V = \{x \in \mathbb{C}^n | f(x) = 0\}$ smooth algebraic variety (hypersurface)
- $M = V \cap \mathbb{R}^n$ differentiable submanifold of $\mathbb{R}^n$
- M is an **algebraic manifold**

Euclidean Connection and Second Fundamental Form

For any manifold M, let $\mathcal{T}(M)$ denote the set of smooth vector fields on M; this is the space of smooth sections of the tangent bundle TM. For $M \subset \mathbb{R}^n$, let $\mathcal{N}(M)$ denote the space of smooth sections of the normal bundle NM. The **Euclidean connection** $\overline{\nabla}$ on $\mathbb{R}^n$ is a map $\overline{\nabla} : \mathcal{T}(\mathbb{R}^n) \times \mathcal{T}(\mathbb{R}^n) \to \mathcal{T}(\mathbb{R}^n), (X, Y) \mapsto \overline{\nabla}_X Y$ defined as follows:

$$(\overline{\nabla}_X Y)(p) = \sum_{i=1}^{n} X_i(p) \frac{\partial Y}{\partial x_i}(p).$$

In other words, $\overline{\nabla}_X Y$ is the vector field whose components are the directional derivatives of the components of Y in the direction X.
The **second fundamental form** of M is the map II from $\mathcal{T}(M) \times \mathcal{T}(M)$ to $\mathcal{N}(M)$ given by

$$\mathrm{II}(X, Y) := (\overline{\nabla}_X Y)^{\perp}.$$

Principal Curvatures

Let $M \subset \mathbb{R}^3$ be a surface. Fix a point $p \in M$ and vector fields $X, Y \in \mathcal{T}(M)$ such that $X(p)$ and $Y(p)$ form an orthonormal basis of T_pM. Let $N(p)$ be a unit vector in N_pM. The **principal curvatures** of M at p are the eigenvalues of the symmetric matrix

$$\begin{bmatrix} \mathrm{II}(X,X)(p)\cdot N(p) & \mathrm{II}(X,Y)(p)\cdot N(p) \\ \mathrm{II}(Y,X)(p)\cdot N(p) & \mathrm{II}(Y,Y)(p)\cdot N(p) \end{bmatrix}.$$

If X and Y are selected so that the matrix is diagonal, then $X(p)$ and $Y(p)$ are the **principal directions**, up to a choice of normal vector.

Umbilics

A point $p \in M$ is called an **umbilic** if all of the principal curvatures at p are equal. At an umbilic, the best second-order approximation of the manifold is a sphere.

Theorem (Salmon, 1865)

The degree of the variety of umbilics of a general surface of degree d in $\mathbb{R}^3$ is $10d^3 - 28d^2 + 22d$.

Definition of Critical Curvature

A point $p \in M$ is called a **point of critical curvature** if there exists a principal curvature c at p such that the gradient of c vanishes in the tangent direction of the unit normal bundle.

Equations for Critical Curvature Locus

The following equations define the locus of pairs (x, u) where $x \in M$ and u is a principal direction at x:

$$f(x_1, \ldots, x_n) = 0,$$

$$\nabla f \cdot u = 0,$$

$$\sum_{i=1}^{n} u_i^2 - 1 = 0,$$

$$\lambda^2(\nabla f \cdot \nabla f) - 1 = 0,$$

$$H_f \cdot u + y_1 u + y_2 \nabla f = 0.$$

The curvature is given by $g(x, u, \lambda) = \lambda u^t \cdot H_f \cdot u$. Using the principle of Lagrange multipliers, we intersect the above locus with the locus defined by the vanishing of the minors of a matrix of partial derivatives of the above equations and partial derivatives of g.

Upper Bound for Critical Curvature Degree

Theorem (Breiding-Ranestad-W.)

Let $V \subset \mathbb{R}^3$ be a general algebraic surface of degree d. Then X has isolated complex critical curvature points. An upper bound for their number is given by $\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$.

d	$\frac{1}{8}(2796d^3 - 6444d^2 + 3696d)$	actual number
2	498	18
3	3573	≥ 456
4	11328	≥ 1808

A formula for hypersurfaces in $\mathbb{R}^n$ can be computed using the same process given in our proof. However, we do not have a proof that a general hypersurface in $\mathbb{R}^n$ for $n \geq 4$ has isolated complex critical curvature points.

Future Work

- Formulate systems of polynomial equations for other concepts in differential geometry and determine the degrees of the relevant varieties.

Thank you!