EXERCISES FOR TAGSS: BOTTLENECKS OF ALGEBRAIC VARIETIES

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1. Definitions

As in the colloquial sense of the word, a bottleneck refers to a narrowing of a variety, or a place where it gets closer to self-intersection.

Consider a smooth algebraic variety $X \subset \mathbb{R}^n$. We define $a \perp b$ by $\sum_{i=1}^{n} a_i b_i = 0$ for $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. For a point $x \in X$, let $(T_x X)_0$ denote the embedded tangent space of X translated to the origin. Then the *Euclidean normal space* of X at x is defined as $N_x X = \{z \in \mathbb{R}^n : (z - x) \perp (T_x X)_0\}$.

Definition 1.1. A *bottleneck* of a smooth algebraic variety $X \subset \mathbb{R}^n$ is a pair of distinct points $(x, y) \in X \times X$ such that $\overline{xy} \subseteq N_x X \cap N_y X$, where $\overline{xy}$ is the line spanned by x and y.

We note that bottlenecks are given not only by the narrowest parts of the variety, but also by maximally wide parts of the variety, as our algebraic definition considers all critical points rather than just the minimums.

Definition 1.2. The *narrowest bottleneck distance* ρ of a variety $X \subset \mathbb{R}^n$ is

$$\rho(X) = \min_{(x,y) \text{ a bottleneck}} d(x, y)$$

where $d(x, y)$ is the Euclidean distance of x and y.

2. Equations

We will now describe the *bottleneck locus* in $\mathbb{R}^{2n}$ which consists of the bottlenecks of X. Let $(f_1, \ldots, f_k) \subseteq \mathbb{R}[x_1, \ldots, x_n]$ be the ideal of X. Consider the ring isomorphism $\phi : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[y_1, \ldots, y_n]$ defined by $x_i \mapsto y_i$ and let $f_i' = \phi(f_i)$. Then f_i and f_i' have gradients ∇f_i and $\nabla f_i'$ with respect to $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, respectively. The *augmented Jacobian* J is the following matrix of size $(k + 1) \times n$ with entries in $R = \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$:

$$J = \begin{bmatrix} y - x \\ \nabla f_1 \\ \vdots \\ \nabla f_k \end{bmatrix},$$

where $y - x$ is the row vector $(x_1 - y_1, \ldots, x_n - y_n)$. Let N denote the ideal in R generated by $(f_1, \ldots, f_k)$ and the $(n - \dim(X) + 1) \times (n - \dim(X) + 1)$ minors of J. Then the points (x, y) of the variety defined by N are the points $(x, y) \in X \times X \subset \mathbb{R}^{2n}$ such that $y \in N_x X$. In the same way we define a matrix J' and an ideal $N' \subseteq R$ by replacing f_i with f_i' and ∇f_i with $\nabla f_i'$.

The *bottleneck locus* B is the variety

$$B = V((N + N') : \langle x - y \rangle^\infty) \subset X \times X \subset \mathbb{R}^{2n}.$$

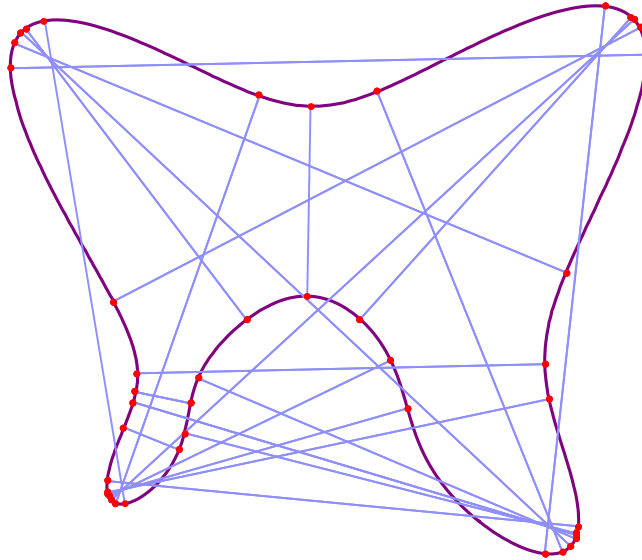The saturation removes the diagonal, as (x, y) is not a bottleneck if $x = y$.

FIGURE 1. The real bottleneck pairs of the butterfly curve. We thank Madeline Brandt for the picture.

3. EXERCISES

3.1. **Sets.** Please work on these exercises in a set S of people, where S satisfies the following conditions.

(1) There exists a member of S who is able to run the Jupyter notebook.
(2) There exists a member of S who is familiar with Macaulay2 (or Singular).
(3) There exists a subset $\{a, b\} \in S$ such that a and b did not meet before this summer school. (Note that this requires the cardinality of S to exceed 1.)

3.2. **Exercises.**

(1) Explore the Jupyter notebook!
(2) Write Macualay2 (or Singular) code to find the bottleneck locus for plane curves, or for varieties of general dimension. (Note that it is possible to create a Julia file directly in M2 to find the variety of a 0-dimensional ideal. See https://www.juliahomotopycontinuation.org/guides/macaulay2/ for code you can copy to M2.)
(3) How many real bottlenecks does a variety have? How many complex bottlenecks does a variety have? What features of the variety does this property depend on? How can you modify the equations for a variety to change this number?
(4) Find equations for varieties with bottlenecks with the following features:
   - many real bottlenecks.
   - narrow bottlenecks.
   - your favorite feature.
(5) Make a beautiful graphic of bottlenecks of your favorite variety.

4. Jupyter Notebook

Sascha Timme created a Jupyter notebook where one can compute and visualize bottlenecks. To access this notebook, install the Julia programming language and then execute the following commands. The third command should open a window in your browser where you can use the notebook.

```
julia> using Pkg; pkg"add https://github.com/saschatimme/TAGSS.git"
julia> using TAGSS
julia> notebooks()
```

5. Further Reading

[BT18] discusses the Julia Homotopy Continuation package used in the Jupyter notebook.

[BW19] computes bottlenecks for plane curves, using both symbolic methods and approximate methods based on a Voronoi decomposition.

[DREW19] gives a formula for the bottleneck degree of an algebraic variety, which under certain conditions corresponds to twice the number of unique complex bottlenecks.

[Ekl18] discusses the numerical algebraic geometry of bottlenecks.

References

[BT18]   Paul Breiding and Sascha Timme. Homotopycontinuation.jl: A package for homotopy continuation in julia. In James H. Davenport, Manuel Kauers, George Labahn, and Josef Urban, editors, *Mathematical Software – ICMS 2018*, pages 458–465, Cham, 2018. Springer International Publishing.

[BW19]   Madeline Brandt and Madeleine Weinstein. Voronoi cells in metric algebraic geometry of plane curves, 2019.

[DREW19] Sandra Di Rocco, David Eklund, and Madeleine Weinstein. The bottleneck degree of algebraic varieties, 2019.

[Ekl18] David Eklund. The numerical algebraic geometry of bottlenecks, 2018.